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Ridge wavelets on the ball[☆]

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Abstract

We present a ridge polynomial wavelet-type system on the unit ball in $\mathbb{R}^d$ such that any continuous function can be expanded with respect to these wavelets. The order of the growth of the degrees of polynomials is optimal. Coefficient functionals are the inner products of the function and the corresponding elements of a "dual wavelet system". The "dual wavelets" is also a ridge polynomial system with the same growth of the degrees of polynomials. The system is redundant.

1. A function $F(x \cdot \theta)$, where $x, \theta \in \mathbb{R}^d$, $x \cdot \theta$ is the inner product and F is an univariate function, is called a wave function (in x) with the wave direction θ. Ridge functions are linear combinations of wave functions. These functions appear naturally in harmonic analysis, special function theory, and in several applications such as tomography and neural networks. Ridge approximation in L_2 was actively studied in the last years by Oskolkov [16–18], Majorov [10], Temlyakov [23], Petrushev [19] and others. Many unexpected phenomena were found. For example, it turned out that the equidistributed wave directions are not necessary optimal even for approximation of radial functions. Logan and Schepp [8] found an orthonormal basis in $L_2(B^2)$ (B^2 is the unit disk) consisting of Chebyshev wave

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polynomials with equispaced wave directions. Their ideas were developed in [16]. Petrushev [19] found a tight frame in $L_2(B^d)$, where B^d is the unit ball in $\mathbb{R}^d$, consisting of wave polynomials, which provides a ridge polynomial expansion for any function $f \in L_2(B^d)$ with the minimal possible growth of the degrees of polynomials.

The goal of this paper is to find ridge polynomial expansions for the space $C(B^d)$. In the one-dimensional case, polynomial expansions of continuous functions on the circle and on the interval was actively studied by many mathematicians for almost forty years. First polynomial basis for $C[a, b]$ was found in 1961 by Foias and Singer [5]. The growth of the degrees of polynomials in this basis was exponential. In 1987, Privalov [20] constructed optimal polynomial bases (regarding the growth of the degrees of polynomials) for the space of continuous functions in both the trigonometric and the algebraic cases. However, his bases were not orthogonal. Optimal trigonometric polynomial orthogonal bases were found due to development of wavelet theory. Offin Oskolkov [15] noted that periodic version of Meyer wavelets provides trigonometric polynomial Schauder basis of optimal (up to a constant factor) growth of the degrees. Lorentz and Sahakian [9] proved that the packets of periodic Meyer wavelets form required bases. Using some generalized shift operators Skopina [21] found a similar wavelet system in $L_2[a, b]$ and proved that the corresponding wavelet packets are optimal polynomial orthogonal bases for the space $C[a, b]$. Though this construction can be realized in any Hilbert space with a polynomial orthogonal basis, it is not clear if the Lebesgue functions of the wavelet Fourier sums are bounded in general. In particular, it is very doubtful that in this way we can provide uniform convergent expansions for continuous functions on the ball and on the sphere. Wavelet-type polynomial systems on the two-dimensional sphere were proposed by Freeden and Schreiner [6]. In contrast to classical wavelet bases these systems are not orthogonal. Moreover, they are even not L_2-bases, and the expansion of an arbitrary function does not converge in L_2, generally speaking. Nevertheless, expansions with respect to such systems are very alike usual wavelet series. In particular, a multiresolution structure is preserved in a certain sense. In [22] Skopina investigated a special cases of Freeden–Schreiner's wavelets and proved that in this case the wavelet expansion of any continuous function uniformly converges to the function. This construction was transferred to the disk due to some special connections between a weighted orthonormal polynomial basis on the disk and the Laplace series. Moreover, since this basis consists of wave polynomials, ridge polynomial expansions for $C(B^2)$ have been found. It was important that $d = 2$ for both the construction and the proofs. In the present paper we will use other ideas to find a similar construction for $d > 2$.

Another construction of ridge wavelets (ridgelets) was proposed by Candes [2–4]. He studied ridgelet expansions of functions in $L_2([0, 1]^d)$. His construction is essentially different from ours.

2. Throughout the paper we consider that a positive integer d is fixed and use the following notations: $x \cdot y = x_1 y_1 + \cdots + x_d y_d$, $|x| = \sqrt{x \cdot x}$ for $x, y \in \mathbb{R}^d$, $\prod_n^d$ is the space of polynomials in d variables of degree at most n, $\mathcal{P}_n := \prod_n^d \ominus \prod_{n-1}^d$, G_n^λ denotes the standard nth Gegenbauer polynomial of order λ,

$$U_n := (h_{n,d/2})^{-1/2} G_n^{d/2},$$

where

$$h_{n,\lambda} = \int_{-1}^{1} (G_n^{\lambda})^2,$$

$B^d = \{x \in \mathbb{R}^d : |x| \leqslant 1\}$ is the unit ball in $\mathbb{R}^d$, $|B^d|$ is the volume of B^d, $S^{d-1} = \partial B^d$ is the unit sphere in $\mathbb{R}^d$, for functions $f, g \in L_2(B^d)$, the inner product is

$$\langle f, g\rangle = \int_{B^d} f(x)\overline{g(x)}\, dx,$$

$$v_n := \frac{(n+1)(n+2)\dots(n+d-1)}{2(2\pi)^{d-1}}.$$

Our arguments will be essentially based on the following results obtained in [19].

The polynomials $U_n(\xi \cdot x)$, $\xi \in S^{d-1}$ are in $\prod_n^d$, $U_n(\xi \cdot x)$, is orthogonal to $\prod_{n-1}^d$ in $L_2(B^d)$, in particularly, if $m \neq n$, then

$$\int_{B^d} U_n(x \cdot \xi) U_m(x \cdot \eta)\, dx = 0, \tag{1}$$

for all $\xi, \eta \in S^{d-1}$, and

$$\int_{B^d} U_n(x \cdot \xi) U_n(x \cdot \eta)\, dx = \frac{U_n(\xi \cdot \eta)}{U_n(1)}. \tag{2}$$

For each $x \in B^d$ and for each $\eta \in S^{d-1}$, we have

$$\int_{S^{d-1}} U_n(x \cdot \xi) U_n(\xi \cdot \eta)\, d\xi = \frac{U_n(1) U_n(x \cdot \eta)}{v_n}. \tag{3}$$

Theorem 1 (*Petrushev [19]*). *Each function $f \in L_2(B^d)$ can be represented uniquely as*

$$f \stackrel{L_2}{=} \sum_{n=0}^{\infty} Q_n(f),$$

where

$$Q_n(f, x) := v_n \int_{S^{d-1}} A_n(f, \xi) U_n(x \cdot \xi)\, d\xi$$

with

$$A_n(f, \xi) = \int_{B^d} f(y) U_n(y \cdot \xi)\, dy.$$

Moreover, the operators Q_n, $n = 0, 1, \dots$, are the orthogonal projectors from $L_2(B^d)$ onto $\mathcal{P}_n$ and the Parseval identity holds

$$\|f\|_{L_2(B^d)}^2 = \sum_{n=0}^{\infty} \|Q_n(f)\|_{L_2(B^d)}^2 = \sum_{n=0}^{\infty} v_n \|A_n\|_{L_2(S^{d-1})}^2.$$

It follows from Theorem 1 that

$$\int_{B^d} dy \int_{S^{d-1}} U_n(y \cdot \xi) U_n(x \cdot \xi)\, d\xi = 0 \tag{4}$$

for all $n > 1$ and for all $x \in B^d$ because the left-hand side of (4) is the orthogonal projection of the function $f \equiv 1$ onto $\mathcal{P}_n$.

Integral representation of Q_n given in Theorem 1 can be rewritten as a discrete sum by using a quadrature formula on S^{d-1}:

$$\int_{S^{d-1}} f(\xi)\, d\xi \cong \sum_{\omega \in \Omega_n} \lambda_\omega f(\omega),$$

where Ω_n is a set of distinct points on S^{d-1}, $\#\Omega_n \asymp n^{d-1}$, $\lambda_\omega \geqslant 0$, which is exact for all spherical polynomials of degree at most $2n$, i.e. for every spherical polynomial S, $\deg S \leqslant 2n$, we have

$$\int_{S^{d-1}} S(\xi)\, d\xi = \sum_{\omega \in \Omega_n} \lambda_\omega S(\omega). \tag{5}$$

Realization of this quadrature is possible for a large class of sets Ω_n due to Theorem 3. It follows from (5) that, for all $x, y \in B^d$,

$$\int_{S^{d-1}} U_n(x \cdot \xi) U_n(y \cdot \xi)\, d\xi = \sum_{\omega \in \Omega_n} \lambda_\omega U_n(x \cdot \omega) U_n(y \cdot \omega), \tag{6}$$

and

$$\begin{aligned} Q_n(f, x) &= v_n \sum_{\omega \in \Omega_n} \lambda_\omega A_n(f, \omega) U_n(x \cdot \omega) \\ &= v_n \int_{B^d} f(y) \sum_{\omega \in \Omega_n} \lambda_\omega U_n(x \cdot \omega) U_n(y \cdot \omega)\, dy. \end{aligned} \tag{7}$$

Furthermore, we have the following equality:

$$\|A_n\|^2_{L_2(S^{d-1})} = \sum_{\omega \in \Omega_n} \lambda_\omega |A_n(f, \omega)|^2.$$

By Theorem 1 and (7), we have a ridge representation for the orthogonal projection of a function $f \in L_2(B^d)$ onto $\mathcal{P}_n$. On the other hand, this projection can be explicitly expressed via an orthonormal polynomial basis for $L_2(B^d)$. Such a basis was found by Xu [24]. This basis consists of algebraic polynomials P_{nk}, $n = 0, 1, \ldots$, $k = 1, \ldots, r_n$, $r_n \asymp n^{d-1}$, the degree of P_{nk} is exactly n. So, we have

$$Q_n(f, x) = \sum_{k=1}^{r_n} \langle f, P_{nk} \rangle P_{nk}(x) = \int_{B^d} f(y) \sum_{k=1}^{r_n} P_{nk}(y) P_{nk}(x)\, dy \tag{8}$$

for all $f \in L_2(B^d)$ and all $x \in B^d$.

Theorem 2 (*Xu [24]*). *The* $(C, d+1)$ *Cesáro means of the Fourier orthogonal series with respect to the system* $\{P_{nk}\}$ *define a positive operator, i.e. for each positive integer N and for all* $x, y \in B^d$, *the following inequality holds*:

$$\sum_{n=0}^{N} \binom{N-n+d+1}{d+1} \sum_{k=1}^{r_n} P_{nk}(y) P_{nk}(x) \geqslant 0. \tag{9}$$

Due to (8), the operator

$$f \longrightarrow \sum_{n=0}^{N} \binom{N-n+d+1}{d+1} \sum_{k=1}^{r_n} \langle f, P_{nk} \rangle P_{nk}$$

coincides with the operator

$$f \longrightarrow \sum_{n=0}^{N} \binom{N-n+d+1}{d+1} Q_n(f).$$

Hence, by Theorem 2, the latter one is also a positive operator, and we have

$$\sum_{n=0}^{N} \binom{N-n+d+1}{d+1} v_n \int_{S^{d-1}} U_n(x \cdot \xi) U_n(y \cdot \xi)\, d\xi \geqslant 0 \tag{10}$$

for each positive integer N and for all $x, y \in B^d$.

3. We need a cubature formula on B^d with nonnegative coefficients. Maybe appropriate formulas are known. Since we could not find them, we present our construction based on the method of iterating quadratures (see, e.g., [14, Chapters 3, 6.4] or [7,13]) and on the following statement summarized the results given in [12] (see also [11, Theorem 2.1]).

Theorem 3. *There exist constants* N_d *and* A_d *depending only on d so that for any finite set* $\{\eta_\ell\}_{\ell \in \Delta}$ *of distinct points* $\eta_\ell \in S^{d-1}$ *and for any positive integer* $N \geqslant N_d$ *satisfying*

$$N \max_{x \in S^{d-1}} \min_{\ell \in \Delta} |x - \eta_\ell| \leqslant A_d,$$

there exist nonnegative weights a_ℓ, $\ell \in \Delta$, *such that*

$$\int_{S^{d-1}} P(x)\, dx = \sum_{\ell \in \Delta} a_\ell P(\eta_\ell)$$

for all $P \in \prod_N^d$.

Due to this theorem, we can assign to each positive integer *j* a set $\{\xi_m^{(j)}\}_{m \in \Delta_j}$ of distinct points $\xi_m^{(j)} \in S^{d-1}$ and a set $\{\alpha_m^{(j)}\}_{m \in \Delta_j}$ of nonnegative weights such that $\sharp\Delta_j \sim 2^{j(d-1)}$ and

$$\int_{S^{d-1}} P(x)\, dx = \sum_{m \in \Delta_j} \alpha_m^{(j)} P(\xi_m^{(j)}) \tag{11}$$

for any $P \in \prod_{2^{j+1}}^{d}$. Iterating this with the Gauss-type quadrature formula

$$\int_{-1}^{1}(1+\tau)^{d-1}\varphi(\tau)\,d\tau \cong \sum_{k=1}^{2^j+1}\gamma_k^{(j)}\varphi\left(\tau_k^{(j)}\right), \tag{12}$$

where $\gamma_k^{(j)} > 0$ and $\tau_k^{(j)}$, $k = 1, \ldots, 2^j + 1$, are the zeros of the Jacobi polynomial $J_{2^j+2}^{(0,d-1)}$ (see, e.g., [1, Chatper 6, Section 2]), we can arrange the following cubature for B^d:

$$\begin{aligned}\int_{B^d} f(x)\,dx &= \int_0^1 \rho^{d-1}\,d\rho \int_{s^{d-1}} f(\rho\xi)\,d\xi \\ &= \frac{1}{2^d}\int_{-1}^{1}(1+\tau)^{d-1}\,d\tau\int_{s^{d-1}} f\left(\frac{\tau+1}{2}\xi\right)\,d\tau \\ &\cong \frac{1}{2^d}\sum_{k=1}^{2^j+1}\sum_{m\in\Delta_j}\gamma_k^{(j)}\alpha_m^{(j)} f\left(\frac{\tau_k^{(j)}+1}{2}\xi_m^{(j)}\right) \\ &= \sum_{k=1}^{2^j+1}\sum_{m\in\Delta_j}\beta_{km}^{(j)} f\left(\frac{\tau_k^{(j)}+1}{2}\xi_m^{(j)}\right).\end{aligned}$$

Redenote the set of points

$$\frac{\tau_k^{(j)}+1}{2}\xi_m^{(j)}, \quad k = 1, \ldots, 2^{j+1}, \;\; m \in \Delta_j$$

by $\{t_\ell^{(j)}\}_{\ell\in D_j}$ and the corresponding factors $\beta_{km}^{(j)}$ by $a_\ell^{(j)}$, $\ell \in D_j$. It is clear that $t_\ell^{(j)} \in B^d$ for all $\ell \in D_j$ and $\sharp D_j \sim 2^{dj}$. Since the quadrature formula (12) is exact on the set $\Pi_{2^{j+1}+1}^{1}$, due to (11), we have

$$\int_{B^d} P(x)\,dx = \sum_{\ell\in D_j} a_\ell^{(j)} P(t_\ell^{(j)}) \tag{13}$$

for any $P \in \Pi_{2^{j+1}+1}^{d}$. Additionally we introduce the set $D_0 := \{0\}$ and put $a_0^{(0)} = |B^d|$.

3. Let

$$h_j(n) = \frac{\binom{2^j-n+d+1}{d+1}}{\binom{2^j+d+1}{d+1}}$$

for $n = 0, \ldots, 2^j$, $h_j(n) = 0$ for $n > 2^j$, and set $g_j(n) = h_j(n) + h_{j-1}(n)$, $\widetilde{g}_j(n) = h_j(n) - h_{j-1}(n)$ for $j = 1, 2, \ldots, n = 0, 1, \ldots, g_0(0) = h_0(0) + 1$, $\widetilde{g}_0(0) = h_0(0) - 1$, $g_0(n) = \widetilde{g}_0(n) = 0$ for $n = 1, 2, \ldots$. For each nonnegative integer j and for each $\ell \in D_{j+1}$, define the wavelet function $\psi_{j\ell}$, the dual wavelet function $\widetilde{\psi}_{j\ell}$ and the scaling function $\varphi_{(j+1)\ell}$ by

$$\psi_{j\ell}(x) = \sum_{n=0}^{\infty} g_j(n)v_n \sum_{\omega\in\Omega_n}\lambda_\omega U_n(t_\ell^{(j+1)}\cdot\omega)U_n(x\cdot\omega),$$

$$\widetilde{\psi}_{j\ell}(x) = \sum_{n=0}^{\infty} \widetilde{g}_j(n) v_n \sum_{\omega\in\Omega_n} \lambda_\omega U_n(t_\ell^{(j+1)} \cdot \omega) U_n(x\cdot\omega),$$

$$\varphi_{(j+1)\ell}(x) = \sum_{n=0}^{\infty} h_j(n) v_n \sum_{\omega\in\Omega_n} \lambda_\omega U_n(t_\ell^{(j+1)} \cdot \omega) U_n(x\cdot\omega).$$

Complete this collection by the function $\varphi_0 = \varphi_{00} \equiv 1/\sqrt{|B^d|}$.

For $f \in C(B^d)$, we will study the convergence of the series

$$\langle f, \varphi_0\rangle\varphi_0 + \sum_{i=0}^{\infty} \sum_{\ell\in D_{i+1}} a_\ell^{(j+1)} \langle f, \widetilde{\psi}_{i\ell}\rangle \psi_{i\ell}. \tag{14}$$

Lemma 4. *For any $f \in C(B^d)$,*

$$\langle f, \varphi_0\rangle\varphi_0 + \sum_{i=0}^{j-1} \sum_{\ell\in D_{i+1}} a_\ell^{(i+1)} \langle f, \widetilde{\psi}_{i\ell}\rangle \psi_{i\ell} = \sum_{\ell\in D_j} a_\ell^{(j)} \langle f, \varphi_{j\ell}\rangle \varphi_{j\ell}. \tag{15}$$

Proof. On the basis of (13) and (6),

$$\begin{aligned}\sum_{\ell\in D_j} a_\ell^{(j)} \langle f, \varphi_{j\ell}\rangle \varphi_{j\ell} = \int_{B^d} dt \int_{B^d} dy\, f(y) &\\ \cdot \sum_{n=0}^{\infty} h_{j-1}(n) v_n \int_{S^{d-1}} U_n(t\cdot\xi) U_n(y\cdot\xi)\, d\xi &\\ \times \sum_{n=0}^{\infty} h_{j-1}(k) v_k \int_{S^{d-1}} U_k(t\cdot\eta) U_k(x\cdot\eta)\, d\eta.&\end{aligned}$$

Using (1), (2) and (3), we derive the right-hand side to

$$\begin{aligned}&\int_{B^d} dy\, f(y) \sum_{n=0}^{\infty} \frac{h_{j-1}^2(n) v_n^2}{U_n(1)} \int_{S^{d-1}} d\xi \int_{S^{d-1}} d\eta\, U_n(\eta\cdot\xi) U_n(y\cdot\xi) U_n(x\cdot\eta)\\ &= \int_{B^d} dy\, f(y) \sum_{n=0}^{\infty} h_{j-1}^2(n) v_n \int_{S^{d-1}} d\eta\, U_n(y\cdot\xi) U_n(x\cdot\eta).\end{aligned}$$

Hence,

$$\sum_{\ell\in D_j} a_\ell^{(j)} \langle f, \varphi_{j\ell}\rangle \varphi_{j\ell} = \sum_{n=0}^{\infty} h_{j-1}^2(n) Q_n(f, x). \tag{16}$$

Similarly,

$$\sum_{\ell\in D_{i+1}} a_\ell^{(i+1)} \langle f, \widetilde{\psi}_{i\ell}\rangle \psi_{i\ell}(x) = \sum_{n=0}^{\infty} \widetilde{g}_i(n) g_i(n) Q_n(f, x). \tag{17}$$

Since $\widetilde{g}_i(n)g_i(n) = h_i^2(n) - h_{i-1}^2(n)$, it follows that

$$\sum_{\ell\in D_i} a_\ell^{(i)}\langle f, \varphi_{i\ell}\rangle\varphi_{i\ell} = \sum_{\ell\in D_{i-1}} a_\ell^{(i-1)}\langle f, \varphi_{(i-1)\ell}\rangle\varphi_{(i-1)\ell} + \sum_{\ell\in D_i} a_\ell^{(i)}\langle f, \widetilde{\psi}_{(i-1)\ell}\rangle\psi_{(i-1)\ell}.$$

Summing these equalities over all $i = 1, \ldots, j$ we obtain (15). □

Lemma 4 shows that the expansions with respect to the systems $\{\varphi_{jk}\}$, $\{\psi_{jk}\}$ have wavelet structure.

Assign to each $\delta \subset D_{j+1}$ the partial sum of (14)

$$\Lambda_{j,\delta}(f) = \langle f, \varphi_0\rangle\varphi_0 + \sum_{i=0}^{j-1}\sum_{\ell\in D_{i+1}} a_\ell^{(i+1)}\langle f, \widetilde{\psi}_{i\ell}\rangle\psi_{i\ell} + \sum_{\ell\in\delta} a_\ell^{(j+1)}\langle f, \widetilde{\psi}_{j\ell}\rangle\psi_{j\ell}.$$

Theorem 5. *For any $f \in C(B^2)$,*

$$\lim_{j\to\infty} \|f - \Lambda_{j,\delta}(f)\|_\infty = 0, \tag{18}$$

where the convergence is uniform over all $\delta \subset D_{j+1}$.

Proof. First we will prove that the operators $\Lambda_{j,\delta}$ taking $C(B^2)$ to $C(B^2)$ are uniformly bounded. By (6), (10) and (13),

$$\begin{aligned}
|\Lambda_{j,\emptyset}(f,x)| &= \left|\sum_{\ell\in D_j} a_\ell^{(j)}\langle f, \varphi_{j\ell}\rangle\varphi_{j\ell}\right| \\
&= \left|\int_{B^d} dt \int_{B^d} dy\, f(y) \sum_{n=0}^{\infty} h_{j-1}(n)v_n \int_{S^{d-1}} U_n(t\cdot\xi)U_n(y\cdot\xi)\,d\xi\right. \\
&\quad \left.\times \sum_{n=0}^{\infty} h_{j-1}(k)v_k \int_{S^{d-1}} U_k(t\cdot\eta)U_k(x\cdot\eta)\,d\eta\right| \\
&\leqslant \|f\|_\infty \int_{B^d} dt \int_{B^d} dy \sum_{n=0}^{\infty} h_{j-1}(n)v_n \int_{S^{d-1}} U_n(t\cdot\xi)U_n(y\cdot\xi)\,d\xi \\
&\quad \times \sum_{k=0}^{\infty} h_{j-1}(k)v_k \int_{S^{d-1}} U_k(t\cdot\eta)U_k(x\cdot\eta)\,d\eta.
\end{aligned}$$

From this, using (4), we obtain

$$\|\Lambda_{j,\emptyset}\| \leqslant v_0^2. \tag{19}$$

Similarly, taking into account that $a_\ell^{(j)} \geqslant 0$, we have

$$\begin{aligned}
&\left| \sum_{\ell\in\delta} a_\ell^{(j+1)} \langle f, \widetilde{\psi}_{j\ell}\rangle \psi_{j\ell}(x) \right| \\
&\quad \leqslant \sum_{\ell\in\delta} a_\ell^{(j+1)} \int_{B^d} dy |f(y)| \sum_{s=j-1}^{j} \sum_{n=0}^{\infty} h_s(n) v_n \\
&\qquad \times \int_{S^{d-1}} U_n(t_\ell^{(j+1)} \cdot \xi) U_n(y\cdot\xi)\, d\xi \\
&\qquad \times \sum_{r=j-1}^{j} \sum_{k=0}^{\infty} h_r(k) v_n \int_{S^{d-1}} U_k(t_\ell^{(j+1)} \cdot \eta) U_k(x\cdot\eta)\, d\eta \\
&\quad \leqslant \|f\|_\infty \sum_{\ell\in D_{j+1}} a_\ell^{(j+1)} \int_{B^d} dy \sum_{s=j-1}^{j} \sum_{n=0}^{\infty} h_s(n) v_n \\
&\qquad \times \int_{S^{d-1}} U_n(t_\ell^{(j+1)} \cdot \xi) U_n(y\cdot\xi)\, d\xi \\
&\qquad \times \sum_{r=j-1}^{j} \sum_{k=0}^{\infty} h_r(k) \int_{S^{d-1}} U_k(t_\ell^{(j+1)} \cdot \eta) U_k(x\cdot\eta)\, d\eta \\
&\quad = \|f\|_\infty \int_{B^d} dt \int_{B^d} dy \sum_{s=j-1}^{j} \sum_{n=0}^{\infty} h_s(n) v_n \int_{S^{d-1}} U_n(t_\ell^{(j+1)} \cdot \xi) U_n(y\cdot\xi)\, d\xi \\
&\qquad \times \sum_{r=j-1}^{j} \sum_{k=0}^{\infty} h_r(k) v_n \int_{S^{d-1}} U_k(t_\ell^{(j+1)} \cdot \eta) U_k(x\cdot\eta)\, d\eta \\
&\quad = 4v_0^2 \|f\|_\infty.
\end{aligned}$$

This and (19) yield $\|\Lambda_{j,\omega}\| \leqslant 5$.

Now, by the Banach–Steinhaus theorem, it suffices to check that (18) holds on the set of polynomials. Let

$$f = \sum_{n=0}^{N} \sum_{k=1}^{r_n} \alpha_{nk} P_{nk}.$$

It follows from (8) and (16) that

$$\begin{aligned}
\Lambda_{j,\emptyset}(f) &= \sum_{n=0}^{N} h_{j-1}^2(n) Q_n(f) \\
&= \sum_{n=0}^{N} h_{j-1}^2(n) \sum_{k=1}^{r_n} \langle f, P_{nk}\rangle P_{nk} \\
&= \sum_{n=0}^{N} h_{j-1}^2(n) \sum_{k=1}^{r_n} \alpha_{nk} P_{nk}.
\end{aligned}$$

Since

$$\lim_{j\to\infty} h_j(n) = 1, \tag{20}$$

whenever n is fixed, we obtain

$$\lim_{j\to\infty} \|f - \Lambda_{j,\emptyset}(f)\|_\infty = 0. \tag{21}$$

Due to Theorem 1,

$$\begin{aligned}\left|\sum_{\ell\in\delta} a_\ell^{(j+1)}\langle f, \widetilde{\psi}_{j\ell}\rangle\psi_{j\ell}\right| &= \left|\sum_{\ell\in\delta} a_\ell^{(j+1)} \sum_{n=0}^{N} \widetilde{g}_j(n) \sum_{k=1}^{r_n} \alpha_{nk} P_{nk}(t_\ell^{(j+1)})\psi_{j\ell}\right| \\ &\leqslant N \max_{0\leqslant n\leqslant N} |h_j(n) - h_{j-1}(n)| \sum_{k=1}^{r_n} |\alpha_{nk}|\|P_{nk}\|_\infty \\ &\quad \times \sum_{\ell\in\delta} |a_\ell^{(j+1)}\psi_{j\ell}(x)|. \end{aligned} \tag{22}$$

By (6), (10) and (13), taking into account the positivity of $a_\ell^{(j+1)}$, we have

$$\begin{aligned}\sum_{\ell\in\delta} |a_\ell^{(j+1)}\psi_{j\ell}(x)| &\leqslant \sum_{\ell\in D_{j+1}} a_\ell^{(j+1)} \left(\left|\sum_{n=0}^{\infty} h_j(n) \int_{S^{d-1}} U_k(t_\ell^{(j+1)}\cdot\eta)U_k(x\cdot\omega)\,d\omega\right|\right. \\ &\quad \left. + \left|\sum_{n=0}^{\infty} h_{j-1}(n) \int_{S^{d-1}} U_k(t_\ell^{(j+1)}\cdot\eta)U_k(x\cdot\omega)\,d\omega\right|\right) \\ &= \int_{B^d} \left(\left|\sum_{n=0}^{\infty} h_j(n) \int_{S^{d-1}} U_k(t\cdot\eta)U_k(x\cdot\omega)\,d\omega\right|\right. \\ &\quad \left. + \left|\sum_{n=0}^{\infty} h_{j-1}(n) \int_{S^{d-1}} U_k(t\cdot\eta)U_k(x\cdot\omega)\,d\omega\right|\right) dt \\ &= 2\int_{B^d} dt = 2|B^d|. \end{aligned}$$

Combining this with (22), we obtain

$$\lim_{j\to\infty} \left\|\sum_{\ell\in\delta} a_\ell^{(j+1)}\langle f, \widetilde{\psi}_{j\ell}\rangle\psi_{j\ell}\right\|_\infty = 0,$$

where the convergence is uniform over all $\delta \subset D_{j+1}$. Due to Lemma 4 and (21), this proves (18). □

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